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THE ENERGY RELEASE RATE AND THE WORK DONE BY THE SURFACE TRACTION IN QUASI-STATIC ELASTIC CRACK GROWTH

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Abstract—This paper presents several expressions for the energy release rate in quasi-static elastic crack growth. A reciprocal type of expression is derived for non-linear materials. As a result, the energy release rate is related to the work done by the surface traction in a cyclic process.

1. INTRODUCTION

In quasi-static crack growth for hyperelastic materials the energy release rate ε is often defined to be the rate of change in potential energy of the entire body[1], or, more generally, by the energy balance for an arbitrary region surrounding a crack tip[2, 3]. It is well known that, for a smooth curved crack, such an ε yields the limiting value of the J-integral as the crack tip is approached, and for a straight crack, the J-integral actually gives ε and is path independent as long as the surface traction vanishes on the crack faces.

Our purpose is to derive alternative expressions for the energy release rate. We begin by defining ε by the energy balance for an arbitrary region surrounding a crack tip. Assuming a special class of the stored energy functions we derive an integral expression of ε, which is reduced to Sanders' reciprocal type of expression[2] for the linear theory. For a general stored energy function, we derive an alternative reciprocal type of expression for ε; we find that it generalizes the well known expression in [1] or [4], which may be evaluated from load versus load-point-displacement relationships for slightly different crack sizes. Finally, we related ε to the work done by the surface tractions in a cyclic process.

We remark that, even for the curved crack, the expressions we derive are path independent and give ε without a limiting process as long as the surface traction vanishes on the crack faces.

2. BASIC EQUATIONS

To fix notation, we consider first a two-dimensional regular body $\mathcal{B}$, which we identify with the regular region of $\mathbb{R}^2$ it occupies in a fixed reference configuration. We assume that the body is homogeneous hyperelastic, so that the (Piola–Kirchhoff) stress $\mathbf{S}(\mathbf{F})$ is the derivative—with respect to the deformation gradient $\mathbf{F}$—of a class C^1 stored energy function $W(\mathbf{F})$:

$$\mathbf{S} = \frac{\partial W(\mathbf{F})}{\partial \mathbf{F}}. \tag{2.1}$$

Here $\mathbf{F} = \nabla \mathbf{y}$ is the gradient of the deformation $\mathbf{y}(\mathbf{x})$ with respect to the material point $\mathbf{x}$ in $\mathcal{B}$.

We will limit our discussion to quasi-static deformations in the absence of body forces: in this instance $\mathbf{S}(\mathbf{x})$ obeys the equilibrium equation

$$\operatorname{div} \mathbf{S} = \mathbf{0}. \tag{2.2}$$

The above equations are appropriate to both the finite and infinitesimal theories of elasticity. In the infinitesimal theory $\mathbf{S}$ is symmetric and W quadratic, but these restrictions are not relevant to what follows.

3. PRELIMINARY DEFINITIONS. ASSUMPTIONS

We assume that $\mathcal{B}$ is bounded and contains an edge crack of negligible thickness, which is allowed to extend with time in $\mathcal{B}$. We note that $\mathcal{B}$ here need not be the entire body. The crack $\mathscr{C}$

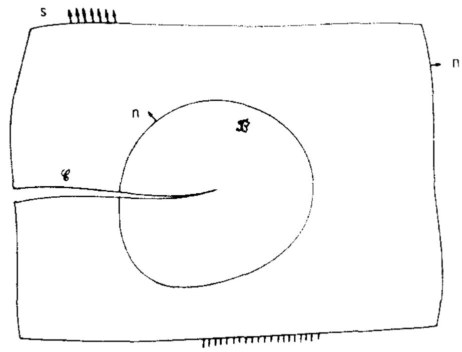

Fig. 1.

is modeled as a smooth, non-intersecting curve (see Fig. 1). Let $\mathscr{C}_l$ denote the crack in B with the length $l(\geq l_0 > 0)$. We then assume that the traction $\mathbf{s}(=\mathbf{Sn})$† and/or the displacement $\mathbf{u}$ on the boundary of the body are changing with a parameter $\alpha(\geq 0)$, so that the fields $\psi(\mathbf{x}, l, \alpha)$ of interest, such as $\mathbf{u}$, $\mathbf{S}$ and W, will be defined at each $\mathbf{x} \in \mathrm{B}\backslash\mathscr{C}_l$, each $l(\geq l_0)$, and $\alpha(\geq 0)$. In what follows, we write $\psi(l, \alpha)$ or simply ψ for $\psi(\mathbf{x}, l, \alpha)$ or for $\psi(\mathbf{F}(\mathbf{x}, l, \alpha))$.

We also assume the following:

(A1) ψ is sufficiently smooth away from the crack and, except at the tip, are continuous up to the crack from either side; in addition, $\psi(l, 0) = 0$ for all $l \geq l_0$;

(A2) $\mathbf{u}$ and $\partial\mathbf{u}/\partial\alpha$ are bounded; W and $\partial W/\partial\alpha$ are integrable uniformly in α [5]; (We note that $\partial W/\partial l$ is not, in general, integrable.)

(A3) the surface traction $\mathbf{s}$ vanishes on the crack and, given any bounded vector field $\mathbf{v}$ on B,

$$\lim_{\delta\to 0}\int_{\partial\Omega_\delta} \mathbf{s}\cdot\mathbf{v}\,\mathrm{d}s = 0,$$

where Ω_δ is the disc of radius δ centered at the crack tip.

4. THE ENERGY RELEASE RATE

The function ε defined by

$$\varepsilon(l, \alpha) = \int_{\partial\mathrm{B}} \mathbf{s}\cdot\frac{\partial\mathbf{u}}{\partial l}\,\mathrm{d}s - \frac{\partial}{\partial l}\int_{\mathrm{B}} W\,\mathrm{d}a \tag{3.1}$$

is called the *energy release rate.*

We first consider a special class of materials with

$$W = \frac{m}{2}\mathbf{S}\cdot\mathbf{F} \qquad m\text{: positive integer.} \tag{3.2}$$

In view of (A1)–(A3), we may use the divergence theorem (see lemma 3 in [5]) to get

$$\int_{\mathrm{B}} \mathbf{S}\cdot\mathbf{F}\,\mathrm{d}a = \int_{\partial\mathrm{B}} \mathbf{s}\cdot\mathbf{y}\,\mathrm{d}s. \tag{3.3}$$

Introducing (3.2) into (3.1) and using (3.3), we have the following

THEOREM 1

$$\varepsilon(l, \alpha) = \int_{\partial\mathrm{B}} \left\{\left(1 - \frac{m}{2}\right)\mathbf{s}\cdot\frac{\partial\mathbf{y}}{\partial l} - \frac{m}{2}\frac{\partial\mathbf{s}}{\partial l}\cdot\mathbf{y}\right\}\mathrm{d}s. \tag{3.4}$$

†The letter $\mathbf{n}$ designates the outward unit normal.

Assuming the infinitesimal linear theory, we can reduce (3.4) to Sanders' reciprocal type of expression[2].

For the general stored energy function W, we have

$$W(l,\alpha)-W(l_0,0)=\int_{(l_0,0)}^{(l,\alpha)} \mathbf{S}(\lambda,\beta)\cdot\frac{\partial \mathbf{H}(\lambda,\beta)}{\partial\lambda}\,\mathrm{d}\lambda+\mathbf{S}(\lambda,\beta)\cdot\frac{\partial \mathbf{H}(\lambda,\beta)}{\partial\beta}\,\mathrm{d}\beta, \tag{3.5}$$

where $\mathbf{H}=\mathbf{F}-\mathbf{1}$ and the integral is clearly path independent in (l,α) space. Thus, noting that $\partial W/\partial l$ $(=\mathbf{S}\cdot(\partial\mathbf{H}/\partial l))$ is not, in general, integrable on $\mathbf{B}\backslash\mathscr{C}_l$, we evaluate (3.5) by taking a special path; such as $(l_0,0)\to(l,0)\to(l,\alpha)$ (see Fig. 2), so that we have

$$W(l,\alpha)=\int_0^\alpha \mathbf{S}(l,\beta)\cdot\frac{\partial \mathbf{H}(l,\beta)}{\partial\beta}\,\mathrm{d}\beta\left(=\int_0^\alpha \frac{\partial W(l,\beta)}{\partial\beta}\,\mathrm{d}\beta\right),$$

since, by (A1), $W(l_0,0)=0$ and $\mathbf{S}(l,0)=\mathbf{0}$ for all $l\geq l_0$. Then, in view of (A1)–(A3), we may use the divergence theorem to get

$$\int_{\mathrm{B}} W(l,\alpha)\,\mathrm{d}a=\int_{\mathrm{B}}\int_0^\alpha \mathbf{S}(l,\beta)\cdot\frac{\partial \mathbf{H}(l,\beta)}{\partial\beta}\,\mathrm{d}\beta\,\mathrm{d}a=\int_{\partial\mathrm{B}}\int_0^\alpha \mathbf{s}(l,\beta)\cdot\frac{\partial \mathbf{u}(l,\beta)}{\partial\beta}\,\mathrm{d}\beta\,\mathrm{d}s. \tag{3.6}$$

Introducing (3.6) into (3.1), we have

THEOREM 2

$$\varepsilon(l,\alpha)=\int_{\partial\mathrm{B}}\int_0^\alpha\left(\frac{\partial\mathbf{s}}{\partial\beta}\cdot\frac{\partial\mathbf{u}}{\partial l}-\frac{\partial\mathbf{s}}{\partial l}\cdot\frac{\partial\mathbf{u}}{\partial\beta}\right)\mathrm{d}\beta\,\mathrm{d}s. \tag{3.7}$$

This expression of ε shows that the energy release rate may be evaluated from traction versus traction-point-displacement relationships for slightly different crack sizes.

Remark 1. If B does not contain the crack, using the divergence theorem on smooth fields in (3.7),

$$\varepsilon(l,\alpha)=\int_{\mathrm{B}}\int_0^\alpha\left(\frac{\partial\mathbf{S}}{\partial\beta}\cdot\frac{\partial\mathbf{H}}{\partial l}-\frac{\partial\mathbf{S}}{\partial l}\cdot\frac{\partial\mathbf{H}}{\partial\beta}\right)\mathrm{d}\beta\,\mathrm{d}a=0,$$

since $\mathrm{d}W=\mathbf{S}\cdot(\partial\mathbf{H}/\partial l)\,\mathrm{d}l+\mathbf{S}\cdot(\partial\mathbf{H}/\partial\beta)\,\mathrm{d}\beta$ is a total differential. Then $\varepsilon(l,\alpha)=0$ in (3.7) formally gives—for the non-linear material—an expression similar to the reciprocal theorem.

Remark 2. Let $\mathbf{e}$ be a unit vector which points to the direction of crack extending. Then the J-integral defined by

$$J=\int_{\partial\mathrm{B}}(W\mathbf{e}\cdot\mathbf{n}-\mathbf{s}\cdot\nabla\mathbf{u}\mathbf{e})\,\mathrm{d}s$$

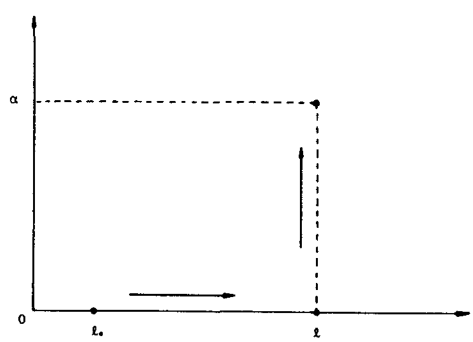

Fig. 2.

is not, in general, path independent unless the crack is straight, since $\mathbf{e} \cdot \mathbf{n} = 0$ on $\mathscr{C}_l$ $\varepsilon(l, \alpha)$ in (3.4) and (3.7) are on the other hand, path independent even if the crack is curved as long as $\mathbf{s} = \mathbf{0}$ on $\mathscr{C}_l$ (see (3.1) with (2.1) and (2.2)).

Suppose $\mathbb{B}$ is the entire body and let

$$\mathbf{s} = \hat{p}(\mathbf{x}, \beta, l)\mathbf{m}, \ \hat{\delta}(\mathbf{x}, \beta, l) = \mathbf{u} \cdot \mathbf{m}, \tag{3.8}$$

where $\mathbf{m}$ is a constant unit vector, so that (3.7) becomes

$$\varepsilon(l, \alpha) = \int_{\partial\mathbb{B}} \int_0^{\alpha} \left(\frac{\partial \hat{p}}{\partial \beta} \frac{\partial \hat{\delta}}{\partial l} - \frac{\partial \hat{p}}{\partial l} \frac{\partial \hat{\delta}}{\partial \beta} \right) \mathrm{d}\beta \, \mathrm{d}s. \tag{3.9}$$

Assume then that $\hat{p}$ is independent of l and let $\bar{p} = \hat{p}(\mathbf{x}, \beta)$ and $p = \hat{p}(\mathbf{x}, \alpha)$. Also assume the existence of the inverse $\hat{\beta} = \hat{p}^{-1}$ for each $\mathbf{x} \in \partial\mathbb{B}$ and define $\tilde{\delta}(\bar{p}, l) = \hat{\delta}(\mathbf{x}, \hat{\beta}(\bar{p}), l)$.
Then (3.9) yields

COROLLARY 1

$$\varepsilon = \int_{\partial\mathbb{B}} \int_0^{p} \frac{\partial \tilde{\delta}(\bar{p}, l)}{\partial l} \mathrm{d}\bar{p} \, \mathrm{d}s.$$

Similarly, exchanging the letter "s and p" for "$\mathbf{u}$ and δ" in (3.8) and in the discussions following (3.9), we have

COROLLARY 2

$$\varepsilon = -\int_{\partial\mathbb{B}} \int_0^{\delta} \frac{\partial \tilde{p}(\bar{\delta}, l)}{\partial l} \mathrm{d}\bar{\delta} \, \mathrm{d}s.$$

The expressions in the above Corollaries, excluding the integral on $\partial\mathbb{B}$, are well known as an alternative valid definition for the J-integral, which may be evaluated from load versus load-point-displacement relationship for slightly different crack sizes (see [1] or [4]).

Let $\alpha = \hat{\alpha}(l)$ be continuous and piecewise smooth for $l \geq l_0$. Then, for $l_0 \leq l^* < l$, (3.7) yields

$$\varepsilon = \frac{\mathrm{d}}{\mathrm{d}l} \int_{\partial\mathbb{B}} \left\{ \int_{l^*}^{l} \int_0^{\hat{\alpha}(\lambda)} \left(\frac{\partial \mathbf{s}}{\partial \beta} \cdot \frac{\partial \mathbf{u}}{\partial \lambda} - \frac{\partial \mathbf{s}}{\partial \lambda} \cdot \frac{\partial \mathbf{u}}{\partial \beta} \right) \mathrm{d}\beta \, \mathrm{d}\lambda \right\} \mathrm{d}s,$$

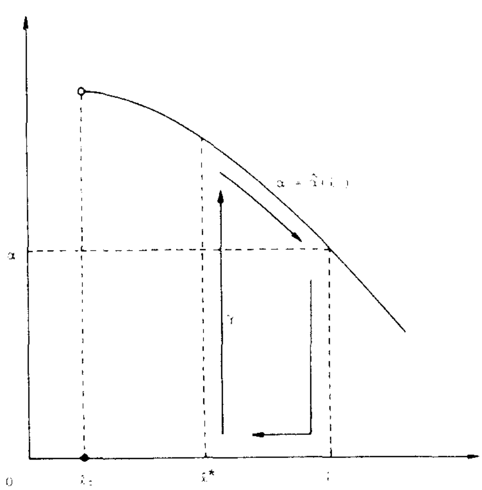

Fig. 3.

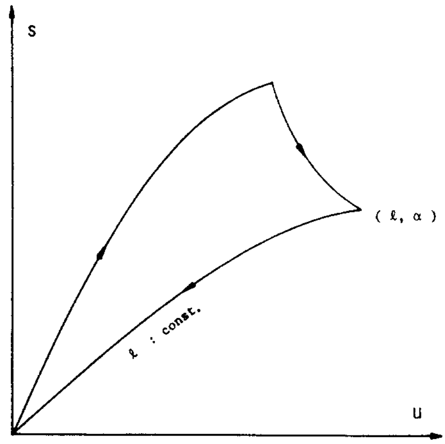

Fig. 4.

so that we may apply Green's theorem to the region in (l, α) space; the result is

THEOREM 3

$$\varepsilon = \frac{\mathrm{d}}{\mathrm{d}l}\int_{\partial B}\left(\int_{\gamma} \mathbf{s}\cdot \mathrm{d}\mathbf{u}\right)\mathrm{d}s,$$

where $\mathrm{d}\mathbf{u} = (\partial \mathbf{u}/\partial\lambda)\,\mathrm{d}\lambda + (\partial\mathbf{u}/\partial\beta)\,\mathrm{d}\beta$ and γ denotes a piecewise smooth closed path such as Fig. 3.

Theorem 3 shows that the (total) released energy from the initial state $(l_0, 0)$ to (l, α) equals the work done by the traction vector in a cyclic process, starting, $\mathbf{s} = \mathbf{0}$, which ends up with the unloading process such that α tends to zero with l constant† (see Fig. 4).

†See the similar discussions of Gurney and Hunt[6], Gurney and Ngan[7] and Burns *et al.*[8], where the discussions are confined to the load-displacement relation and are not concerned with the singularities near the tip.

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